DECEMBER, 1886.

ANNALS OF MATHEMATICS.

EDITED BY

ORMOND STONE AND WILLIAM M. THORNTON.

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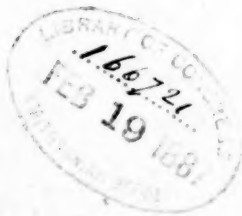

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ERRATA.

p. 36, lines 9, 10, and 11, read $d\phi$ instead of $d\rho$.

p. 36, line 13, in second term of denominator in third term, read $\frac{e^2}{(d-r)^2}$ instead of $\frac{e}{(d-r)^2}$.

p. 60, first and last formula, for r read r_1.

p. 61, first formula should be

$$b = \frac{1-\chi}{2\chi} \cdot \frac{r_1}{1 - \frac{r_1}{2a\chi}}$$

$$= \frac{1-\chi}{2\chi} r_1 \text{ when the orbit of the comet is a parabola.}$$

p. 62, formula for cubic equation, for $a_0 a_1^{\frac{3}{2}}$ read $a_0^{\frac{3}{2}} a_1$, and for tan ϕ read tan $\frac{1}{2}\phi$.

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ON THE FORM AND POSITION OF THE SEA-LEVEL AS DEPENDENT ON SUPERFICIAL MASSES SYMMETRICALLY DISPOSED WITH RESPECT TO A RADIUS OF THE EARTH'S SURFACE.

By MR. R. S. WOODWARD, Washington, D. C.

[CONTINUED FROM VOL. II, PAGE 103.]

9. Some special values of the integrals (22)–(25), and the corresponding potentials (20) and (21), are worth deriving. These values are

a. For a point of the disturbed surface at the centre of the disturbing mass $\alpha = 0$, and (24) gives

$$I_1 = 2 \sin \tfrac{1}{2}\beta \int_0^{\frac{1}{2}\pi} d\gamma_1 = \pi \sin \tfrac{1}{2}\beta; \tag{26}$$

and the corresponding potential is

$$V_1 = 4 r_0 h \rho \pi \sin \tfrac{1}{2}\beta. \tag{27}$$

b. For a point at the border of the disturbing mass $\alpha = \beta$, and hence from (24) or (25)

$$I_1 = I_2 = 2 \arcsin \left[\sin \tfrac{1}{2}\beta \sin \gamma \right]_{\gamma=0}^{\gamma=\frac{1}{2}\pi} = \beta, \tag{28}$$

a result which is reached quite as readily from (22) or (23). The corresponding value of the potential is

$$V_1 = V_2 = 4 r_0 h \rho \beta. \tag{29}$$

c. For a point of the disturbed surface 180° from the centre of the disturbing mass $\alpha = \pi$, and (25) gives

$$I_2 = 2 \int_0^{\frac{1}{2}\pi} \frac{\sin^2 \tfrac{1}{2}\beta \cos^2 \gamma_2 \, d\gamma_2}{1 - \sin^2 \tfrac{1}{2}\beta \sin^2 \gamma_2}$$

$$= 2\int_0^{\frac{1}{2}\pi} \frac{(1 - \cos^2 \frac{1}{2}\beta)\, d\gamma_2}{1 + \cos^2 \frac{1}{2}\beta \tan^2 \gamma_2} \qquad (30)$$

$$= \pi - 2 \cos \tfrac{1}{2}\beta \arctan \left[\tan (\cos \tfrac{1}{2}\beta \tan \gamma_2) \right]_0^{\frac{1}{2}\pi}$$

$$= 2\pi \sin^2 \tfrac{1}{4}\beta;$$

and the potential is

$$V_2 = 8 r_0 h \rho \pi \sin^2 \tfrac{1}{4}\beta. \qquad (31)$$

d. Suppose $\beta = \pi$; in other words, let the attracting mass cover the whole sphere; then (24) gives

$$I_1 = \pi, \qquad (32)$$

and we have the following well known approximate value for the potential of a spherical shell for a point on its exterior or interior surface, viz:—

$$4 r_0 h \rho \pi. \qquad (33)$$

This result follows also from (27) or (29), if we make $\beta = \pi$.

10. In deriving the expressions (20) and (21) for the potential of the disturbing mass, it was assumed that a sufficient degree of approximation is attained if quantities of the orders u/r_0, v/r_0, and upwards are neglected. The grounds of this assumption need to be examined with some care. For this purpose we shall derive the exact expressions, in form at least, for the potential of the disturbing mass at its centre, at its border, and at 180° from its centre. A comparison of these exact values with the approximate values given in (27), (29), and (31) will show the order of approximation of (20) and (21).

We will first write down the exact expression for the potential of a spherical shell of uniform thickness and density, for a point within its bounding surfaces. This expression will be useful as a check on formulas relating to partial shells. Let the radius of the interior surface of the complete shell be r_0, the uniform thickness h, the uniform density ρ, and the distance of the attracted point from the centre of the shell $r_0 + v$; then the potential is *

$$V = 4 r_0 h \rho \pi \left[1 + \frac{h}{2(r_0 + v)} - \frac{v^2}{2(r_0 + v)h} + \frac{hv}{2(r_0 + v) r_0} - \frac{v^3}{6(r_0 + v) r_0 h} \right]. \qquad (34)$$

11. Let the notation be the same as that used heretofore; i. e. let r be the

* $2\rho\pi \left[(r_0 + h)^2 - \frac{1}{3}(r_0 + v)^2 - \frac{2}{3}\frac{r_0^3}{r_0 + v} \right]$.

See Price's *Calculus*, Vol. III. p. 299.

radius-vector of any point of the attracting mass, r' that of any point of the disturbed surface; ρ the density, h the thickness, and β the angular radial extent of the mass; and θ the angular distance between the attracted and attracting points, the origin being at the centre of the sphere. For points of the disturbed surface lying above the undisturbed surface, r will be less than $r' = r_0 + v$ over the range r_0 to r', and r will be greater than r' over the range r' to $r_0 + h$. Bearing these facts in mind, it follows that the exact value of the potential of the disturbing mass, for the point where its axis pierces the disturbed surface, is

$$V = 2\rho\pi \int_{r_0}^{r_0+v} r^2 dr \int_0^{\beta} \frac{\sin\theta\, d\theta}{\sqrt{(r^2 + r'^2 - 2rr'\cos\theta)}}$$
$$+ 2\rho\pi \int_{r_0+v}^{r_0+h} r^2 dr \int_0^{\beta} \frac{\sin\theta\, d\theta}{\sqrt{(r^2 + r'^2 - 2rr'\cos\theta)}}$$
$$= 2\rho\pi \int_{r_0}^{r_0+v} \left[\sqrt{[4rr'\sin^2\tfrac{1}{2}\beta + (r'-r)^2]} - (r'-r)\right] \frac{r\,dr}{r'}$$
$$+ 2\rho\pi \int_{r_0+v}^{r_0+h} \left[\sqrt{[4rr'\sin^2\tfrac{1}{2}\beta + (r-r')^2]} - (r-r')\right] \frac{r\,dr}{r'}.$$

By expansion, integration, and reduction this expression gives, to terms of the first order inclusive,

$$V = 4r_0 h\rho\pi \sin\tfrac{1}{2}\beta \left(1 + \frac{3h\sin\tfrac{1}{2}\beta - h}{4(r_0+v)\sin\tfrac{1}{2}\beta} + \frac{v(1-\sin\tfrac{1}{2}\beta)}{2(r_0+v)\sin\tfrac{1}{2}\beta} - \frac{v^2}{2(r_0+v)h\sin\tfrac{1}{2}\beta}\right). \tag{35}$$

This equation, if we make $\beta = \pi$, agrees with (34), as it should, to terms of the second order. It differs from (27) by terms which must be small unless h and v are very large. In one of the most important applications discussed by the writer, $h = 10\,000$ feet, $v = 3000$ feet and $\beta = 38°$. With these values, since r_0 is in round numbers 21 000 000 feet, the quantity in the brackets of (35) differs from unity by less than 1/10 000. If $\beta = 60°$, which is (as shown in the sequel) the angular extent of mass required to produce the maximum elevation of the disturbed surface at the centre of the mass, the quantity within the brackets of (35) exceeds unity by less than 1/8000, using the above value of h and v.

12. Similarly, if the attracted point be at the border of the attracting mass, the exact value of the potential is

$$V = 2\rho \int_{r_0}^{r_0+v} \frac{r}{r'}\, dr \int_0^{\frac{1}{2}\pi} \sqrt{[(r'-r)^2 + 4rr'\sin^2\tfrac{1}{2}\theta_2]}\, .\, d\lambda$$

$$+ 2\rho \int_{r_0+v}^{r_0+h} \frac{r}{r'} dr \int_0^{\frac{1}{2}\pi} 1 \Big/ [(r-r')^2 + 4rr' \sin^2 \tfrac{1}{2}\theta_2] \, . \, d\lambda$$

$$- \rho\pi \int_{r_0}^{r_0+v} (r'-r) \frac{r}{r'} dr - \rho\pi \int_{r_0+v}^{r_0+h} (r-r') \frac{r}{r'} dr,$$

in which

$$\sin^2 \tfrac{1}{2}\theta_2 = \frac{\sin^2 \beta \cos^2 \lambda}{1 - \sin^2 \beta \sin^2 \lambda}.$$

Introducing this value, the first of the above integrals relative to λ becomes

$$(r'+r) \int_0^{\frac{1}{2}\pi} \sqrt{\left\{ \frac{\sin^2 \beta \cos^2 \lambda + \left(\frac{r'-r}{r'+r}\right)^2 \cos^2 \beta}{1 - \sin^2 \beta \sin^2 \lambda} \right\}} \, . \, d\lambda .$$

Since the numerator of the element-function in this integral is greater than $\sin \beta \cos \lambda$, and less than $\sin \beta \cos \lambda + \frac{r'-r}{r'+r} \cos \beta$, the value of the integral lies between

$$(r'+r) \beta \quad \text{and} \quad (r'+r) \beta \left\{ 1 + \frac{r'-r}{r'+r} \frac{\cos \beta}{\beta} \int_0^{\frac{1}{2}\pi} \frac{d\lambda}{1 (1 - \sin^2 \beta \sin^2 \lambda)} \right\} .$$

Therefore, if Q_1 denote a suitable factor, the exact value of the integral may be written thus: —

$$(r'+r) \beta \left\{ 1 + Q_1 \frac{r'-r}{r'+r} \right\} .$$

Likewise represent the exact value of the second integral relative to λ by

$$(r+r') \beta \left\{ 1 + Q_2 \frac{r-r'}{r+r'} \right\} .$$

Then, to terms of the first order, inclusive, the above expression for the potential becomes

$$V = 4 r_0 h \rho \beta \left\{ \begin{array}{l} 1 + \dfrac{h(3-Q_2)}{4(r_0+v)} - \dfrac{v(1+Q_2)}{2(r_0+v)} + \\ \dfrac{v^2(Q_1+Q_2)}{4 r_0 h} - \dfrac{\pi}{\beta} \cdot \dfrac{h^2 - 2hv + 2v^2}{8(r_0+v)h} \end{array} \right\} . \qquad (36)$$

The first term of this agrees with (29). The factors Q_1 and Q_2 lie between 0 and the elliptic of the first species

$$\frac{\cos \beta}{\beta} \int_0^{\frac{1}{2}\pi} \frac{d\lambda}{1 (1 - \sin^2 \beta \sin^2 \lambda)} .$$

When $\beta = \frac{1}{2}\pi$, or when the attracting stratum covers a whole hemisphere, $Q_1 = Q_2 = 0$, and (36) agrees, as it should, with the half of (34) to terms of the second order; and in this case if $h = 10\,000$ feet and $v = 3000$ feet, the quantity in the parenthesis of (36) exceeds unity by 1 / 4000. When $\beta < \frac{1}{2}\pi$, Q_1 and Q_2 will exceed 0, and the degree of approximation of (36) will be somewhat greater than in the case just discussed.

13. For a point of the disturbed surface 180° from the centre of the disturbing mass, the exact value of the potential is

$$V = 2\rho\pi \int_{r_0}^{r_0+h} r^2 dr \int_0^{\beta} \frac{\sin\theta\, d\theta}{\sqrt{(r^2 + r'^2 + 2rr'\cos\theta)}}$$

$$= 2\rho\pi \int_{r_0}^{r_0+h} [r + r' - \sqrt{(r^2 + r'^2 + 2rr'\cos\beta)}]\, \frac{r}{r'}\, dr.$$

Expanding, integrating with respect to r, and reducing, there results, to terms of the first order inclusive,

$$V = 8\, r_0 h \rho\pi \sin^2 \tfrac{1}{4}\beta \left(1 + \frac{3h - 2v}{4\,(r_0 + v)} \right). \qquad (37)$$

The first term of this agrees with (31). Using the above values of r_0 and h, and $v = -1000$ feet (v being here intrinsically negative), the factor in the brackets of (37) exceeds unity by 1 / 2600.

14. The preceding expressions for the potential of the attracting mass, namely, V as defined by equations (20) and (21), are sufficient for most of the cases they were designed to meet. They possess the obvious advantage of a compact integral form. For some purposes, however, it is desirable to have V expressed in a series of spherical harmonics, or Laplace's functions. We may thereby arrive at equations (20) and (21), by a process differing from that followed in § 7, and establish a useful harmonic development of the elliptic integrals I_1 and I_2 of equations (22) to (25).

Expressions fulfilling the present requirements may be derived from equation (10) by expanding D^{-1} in a series of ascending powers of $\frac{r}{r'}$ or $\frac{r'}{r}$. Thus from equation (9) we have

$$\frac{1}{D} = \frac{1}{r'}\left[P_0 + P_1 \left(\frac{r}{r'}\right) + P_2 \left(\frac{r}{r'}\right)^2 + \ldots \right], \text{ when } r < r';$$

$$\frac{1}{D} = \frac{1}{r}\left[P_0 + P_1 \left(\frac{r'}{r}\right) + P_2 \left(\frac{r'}{r}\right)^2 + \ldots \right], \text{ when } r > r'.$$

In these equations $P_0, P_1, P_2, \ldots$ are Laplace's co-efficients of the zero, first,

second, etc. order respectively. They are functions of the angular co-ordinates θ, θ', λ, λ' only.

Taking now the centre of the sphere of reference as the origin of co-ordinates, and supposing the line from which θ and θ' are reckoned to pass through the centre of the attracting mass, we shall have for a mass of uniform thickness h, all integrations in (10) independent. For that part of the disturbed surface which lies above the undisturbed surface, $r' = r_0 + v$ will fall between the extreme values of r, which are r_0 and $r_0 + h$; and hence

$$r < r' \text{ for values between } r_0 \text{ and } r_0 + v,$$
$$r > r' \text{ for values between } r_0 + v \text{ and } r_0 + h.$$

For that part of the disturbed surface lying below the undisturbed surface, $r > r'$. In both cases the limits of θ are 0 and the angular radius β of the attracting mass; and the limits of λ are 0 and 2π. Therefore for that part of the disturbed surface lying above the undisturbed surface, equation (10) gives

$$V = \rho \int_0^{\beta} \sin\theta\, d\theta \int_0^{2\pi} d\lambda \int_{r_0}^{r_0+v} \left[P_0 + P_1 \left(\frac{r}{r'}\right) + P_2 \left(\frac{r}{r'}\right)^2 + \ldots \right] \frac{r^2 dr}{r'}$$
$$+ \rho \int_0^{\beta} \sin\theta\, d\theta \int_0^{2\pi} d\lambda \int_{r_0+v}^{r_0+h} \left[P_0 + P_1 \left(\frac{r'}{r}\right) + P_2 \left(\frac{r'}{r}\right)^2 + \ldots \right] r dr. \quad (38)$$

Likewise, for that part of the disturbed surface which lies below the undisturbed surface, equation (10) gives

$$V = \rho \int_0^{\beta} \sin\theta\, d\theta \int_0^{2\pi} d\lambda \int_{r_0}^{r_0+h} \left[P_0 + P_1 \left(\frac{r'}{r}\right) + P_2 \left(\frac{r'}{r}\right)^2 + \ldots \right] r dr. \quad (39)$$

For brevity, let the integrals with respect to r in (38) be denoted as follows: —

$$J_i = \int_{r_0}^{r_0+v} \left(\frac{r}{r'}\right)^i \frac{r^2 dr}{r'} = \frac{(r_0+v)^{i+3} - r_0^{i+3}}{(i+3)(r_0+v)^{i+1}}, \quad (40)$$

$$J'_i = \int_{r_0+v}^{r_0+h} \left(\frac{r'}{r}\right)^i r dr = \frac{(r_0+v)^i}{i-2} \left[(r_0+v)^{-(i-2)} - (r_0+h)^{-(i-2)} \right]. \quad (41)$$

The ambiguous form assumed by (41) when $i = 2$ is easily shown to be equal to

$$(r_0+v)^2 \log_e \frac{r_0+h}{r_0+v}.$$

Similarly, let the integrals in (39) with respect to r be denoted thus: —

$$J''_i = \frac{(r_0 + v)^i}{i-2}\left[r_0^{-(i-2)} - (r_0 + h)^{-(i-2)}\right]. \qquad (42)$$

Substituting these equivalents in (38) and (39), they become respectively

$$V = \rho \int_0^\beta \sin\theta\, d\theta \int_0^{2\pi} \left[(J_0 + J'_0) P_0 + (J_1 + J'_1) P_1 + \ldots\right] d\lambda, \qquad (43)$$

$$V = \rho \int_0^\beta \sin\theta\, d\theta \int_0^{2\pi} \left[J''_0 P_0 + J''_1 P_1 + \ldots\right] d\lambda. \qquad (44)$$

Now it is known from the theory of spherical harmonics * that the general value of P_i is

$P_i = f_i(\cos\theta).f_i(\cos\theta') +$ terms multiplied by cosines of multiples of $(\lambda - \lambda')$.

But since

$$\int_0^{2\pi} \cos i(\lambda - \lambda')\, d\lambda = 0,$$

we have to deal only with the first term of P_i. If we write for brevity $\mu = \cos\theta$, the function of $\cos\theta$ or $\cos\theta'$ involved in this first term, is defined thus: —

$$f_i(\cos\theta) = f_i(\mu) = \frac{1}{2^i \cdot 1 \cdot 2 \cdot 3 \ldots i} \cdot \frac{d^i(\mu^2 - 1)^i}{d\mu^i}.$$

The following important relation exists between any three consecutive values of $f_i(\mu)$, viz.: —

$$f_i(\mu) = \frac{2i-1}{i}\mu f_{i-1}(\mu) - \frac{i-1}{i} f_{i-2}(\mu). \qquad (45)$$

The remaining integrals in (43) and (44) are therefore of the form

$$f_i(\cos\theta') \int_0^\beta f_i(\cos\theta) \sin\theta\, d\theta \int_0^{2\pi} d\lambda = 2\pi f_i(\mu') \int_{\cos\beta}^1 f_i(\mu)\, d\mu.$$

Let

$$F_i(\beta) = \int_{\cos\beta}^1 f_i(\mu)\, d\mu. \qquad (46)$$

The known value of this integral is

$$F_i(\beta) = \frac{1}{2i+1}\left[f_{i-1}(\cos\beta) - f_{i+1}(\cos\beta)\right]. \qquad (47)$$

The values of V in (43) and (44) may now be written thus, replacing θ' by

* See Heine, *Handbuch der Kugelfunctionen*, or Todhunter's *Treatise on Laplace's, Lamé's, and Bessel's Functions.*

by α, θ' or α being the angular distance of the attracted point from the centre of the attracting mass: —

$$V = 2\rho\pi \sum_{i=0}^{i=\infty} (J_i + J'_i) f_i(\cos\alpha) F_i(\beta), \tag{48}$$

for points of disturbed surface above undisturbed;

$$V = 2\rho\pi \sum_{i=0}^{i=\infty} J''_i f_i(\cos\alpha) F_i(\beta), \tag{49}$$

for points of disturbed surface below undisturbed.

15. The values of V in the last two equations are exact, but their applicability to practical problems is limited by the slow convergence of the series in the second members. In most cases with which we are concerned, r_0, $r_0 + v$ and $r_0 + h$ are nearly equal, and hence the convergence depends almost wholly on the convergence of the functions of α and β.

If we neglect terms of the order h/r_0, v/r_0, and upwards, equations (48) and (49) become identical and equivalent to (20) and (21). To show these facts, we expand the second members of (40), (41), and (42), and obtain, to terms of the second order,

$$J_i + J'_i = r_0 h \left[1 - \frac{1}{2r_0} \left\{ (i-1)h - 2iv + (i + \tfrac{1}{2}) \frac{v^2}{h} \right\} \right] \tag{50}$$

$$J''_i = r_0 h \left[1 - \frac{(i-1)h}{2r_0} + \frac{iv}{r_0} \right]. \tag{51}$$

Hence, to terms of the first order, we have *

$$J_i + J'_i = J''_i = r_0 h,$$

and (48) and (49) become

$$V = 2r_0 h\rho\pi \sum_{i=0}^{i=\infty} f_i(\cos\alpha) F_i(\beta). \tag{52}$$

Now if $\cos\psi = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos(\lambda - \lambda')$,

$$2\pi \sum_{i=0}^{i=\infty} f_i(\cos\alpha) F_i(\beta) = \int_0^{\beta}\int_0^{2\pi} \frac{\sin\theta\, d\theta\, d\lambda}{\sqrt{(1 - 2\cos\psi + 1)}}.$$

But this integral, as shown in § 7, is equivalent to I_1 or I_2 of (22) or (23) according as α is less or greater than β. This establishes the equivalence of (52) with (20)

* This inference from (50) and (51) does not appear to be quite satisfactory. For large values of i, $J_i + J'_i$, and J''_i are less than $r_0 h$; they are each zero for $i = \infty$. The influence of the too great factor $r_0 h$ in the higher terms of (52) is counteracted, however, by the small factors $f_i(\cos\alpha)$ $F_i(\beta)$ in those terms. The order of approximation of (52) is evident from its equivalence with (20) and (21), whose order of approximation is investigated in §§ 10–13.

and (21), and furnishes the following development in polar harmonics of the elliptic integrals in (22) to (25): —

$$I_1 = \int_0^a \sqrt{\left(\frac{\cos p - \cos \beta}{\cos p - \cos \alpha}\right)} \, . \, dp = \tfrac{1}{2}\pi \sum_{i=0}^{i=\infty} f_i(\cos \alpha) \, F_i(\beta),$$

$$\alpha \leqq \beta \qquad (53)$$

$$I_2 = \int_0^\beta \sqrt{\left(\frac{\cos p - \cos \beta}{\cos p - \cos \alpha}\right)} \, . \, dp = \tfrac{1}{2}\pi \sum_{i=0}^{i=\infty} f_i(\cos \alpha) \, F_i(\beta).$$

$$\alpha \geqq \beta$$

16. In order to show the forms of the second members of equations (52) and (53), we give below the first four values of $f_i(\cos \alpha)$ and $F_i(\beta)$. The series may be easily extended by means of the relations in (45) and (47).

$$\begin{aligned}
&f_0(\cos \alpha) = 1, \\
&f_1(\cos \alpha) = \cos \alpha, \\
&f_2(\cos \alpha) = \tfrac{1}{2}(3 \cos^2 \alpha - 1), \\
&f_3(\cos \alpha) = \tfrac{1}{2}(5 \cos^3 \alpha - 3 \cos \alpha);
\end{aligned}$$

$$\begin{aligned}
&F_0(\beta) = 1 - \cos \beta, \\
&F_1(\beta) = \tfrac{1}{2}(1 - \cos^2 \beta), \\
&F_2(\beta) = \tfrac{1}{2}(\cos \beta - \cos^3 \beta), \\
&F_3(\beta) = \tfrac{1}{8}(6 \cos^2 \beta - 5 \cos^4 \beta - 1).
\end{aligned}$$

By means of these values equation (52) may be written thus: —

$$V = 2 r_0 h \rho \pi \left\{ \begin{array}{l} 2 \sin^2 \tfrac{1}{2}\beta + \tfrac{1}{2} \cos \alpha \sin^2 \beta \\ + \tfrac{1}{4}(3 \cos^2 \alpha - 1) \sin^2 \beta \cos \beta \\ + \tfrac{1}{16}(5 \cos^3 \alpha - 3 \cos \alpha)(6 \cos^2 \beta - 5 \cos^4 \beta - 1) \\ + \ . \ . \ . \end{array} \right\}. \qquad (54)$$

17. In case the disturbing mass is as large as the supposed ice mass of the glacial epoch, the attraction of the re-arranged free water on itself may be appreciable. To determine the exact effect of this attraction would be a work of great difficulty even if we had the requisite information, namely, an accurate knowledge of the complicated shapes of the continents and sea-bottom. But we may determine an effect which will in general exceed the probable actual effect by supposing the whole surface of the earth covered with a film of water free to assume the proper form for equilibrium under the given forces. To fit this ideal case, formula (6) may be modified in the following manner: —

Let the potential V in (6) be replaced by $V + \Delta V$, where ΔV is the potential

due to the re-arrangement of the water. Likewise, replace the constant V_0 by $V_0 + \Delta V_0$. Then if Δv denote the corresponding change in v, equation (6) gives

$$v + \Delta v = \frac{3}{4 r_0 \rho_m \pi} (V + \Delta V - V_0 - \Delta V_0). \qquad (55)$$

Now $v + \Delta v$ may be expressed by a series of Laplace's functions (which are in this case polar harmonics), thus: —

$$v + \Delta v = r_0 c_0 (Z_0 + Z_1 + Z_2 + \ldots),$$

in which c_0 is a constant of small numerical value; and hence, denoting the density of sea water by ρ_w, we have, as shown by Laplace*,

$$\Delta V = 4 r_0^2 c_0 \rho_w \pi \left(Z_0 + \tfrac{1}{3} Z_1 + \tfrac{1}{5} Z_2 + \ldots + \frac{1}{2i+1} Z_i + \ldots\right).$$

V has already been expressed in equation (54) in a series of Laplace's functions. Denoting these functions in (54) for brevity by Y_0, Y_1, Y_2, etc.,

$$V = 2 r_0 h \rho \pi (Y_0 + Y_1 + Y_2 + \ldots),$$

Substituting these values of V, ΔV, and $v + \Delta v$ in (55), there results, if we make the obviously permissible substitution,

$$\tfrac{3}{2} h \frac{\rho}{\rho_m} U_0 = \frac{3}{4 r_0 \rho_m \pi} (V_0 + \Delta V_0),$$

the following equation: —

$$\left\{\begin{array}{ll} r_0 c_0 Z_0 \left(1 - \frac{3}{1} \frac{\rho_w}{\rho_m}\right) & - 3h \frac{\rho}{\rho_m} (Y_0 - U_0) \\ + r_0 c_0 Z_1 \left(1 - \frac{3}{3} \frac{\rho_w}{\rho_m}\right) & - 3h \frac{\rho}{\rho_m} Y_1 \\ + r_0 c_0 Z_2 \left(1 - \frac{3}{5} \frac{\rho_w}{\rho_m}\right) & - 3h \frac{\rho}{\rho_m} Y_2 \\ + \ldots & \\ + r_0 c_0 Z_i \left(1 - \frac{3}{2i+1} \frac{\rho_w}{\rho_m}\right) & - 3h \frac{\rho}{\rho_m} Y_i \end{array}\right\} = 0.$$

According to the theory of spherical harmonics, we must have in this equation the sums of the harmonics of the same order separately equal to zero, which

**Mécanique Céleste*, Book III. Chap. 2.

amounts to placing each line in the equation equal to zero. Hence we have

$$v + \Delta v = r_0 c_0 \sum_{i=0}^{i=\infty} Z_i$$

$$= \tfrac{3}{2} h \frac{\rho}{\rho_m} \left\{ \sum_{i=0}^{i=\infty} \frac{Y_i}{1 - \frac{3}{2i+1}\frac{\rho_w}{\rho_m}} - \frac{U_0}{1 - 3\frac{\rho_w}{\rho_m}} \right\}. \qquad (56)$$

This equation expresses the total effect of the disturbing mass in altering the sea-level, v being the effect which would result if the ocean were an infinitely rare fluid, and Δv being the increase over v which would result under the assumed conditions. Obviously v and Δv may be expressed separately. Thus: —

$$\Delta v = \tfrac{3}{2} h \frac{\rho}{\rho_m} \left\{ \sum_{i=0}^{i=\infty} \frac{3 Y_i}{(2i+1)\frac{\rho_m}{\rho_w} - 3} - \frac{3 U_0}{\frac{\rho_m}{\rho_w} - 3} \right\}. \qquad (57)$$

[TO BE CONTINUED.]

NOTES.

8

ON THE LAST CLAUSE OF ART. 140 OF GAUSS'S THEORIA MOTUS.

The clause mentioned is as follows: —

"Calculo examinando sequentes quoque formulae inservire possunt, quarum tamen derivationem non ita difficilem brevitatis caussa supprimimus: —

$$0 = \frac{a \sin(l'' - l')}{R} - \frac{b \sin(l'' - l)}{R'} \cdot \frac{\sin(\delta' - \sigma)}{\sin \delta'} + \frac{\sin(l' - l)}{R''},$$

$$b = \frac{R' \sin \delta'}{R'' \sin \delta''} \cdot \frac{U \cos \beta \cos \beta''}{\sin(AD' - \delta) \sin \varepsilon'};$$

ubi U exprimit quotientem $\dfrac{S}{\sin(\delta' - \sigma)} = \dfrac{T \sin(t + \gamma')}{\cos(\delta' - \sigma)}$ (art. 138. aequ. 10)."

Notwithstanding the words "non ita difficilem," the derivation of these equations is not generally readily seen by the student reading the *Theoria Motus* for the first time.

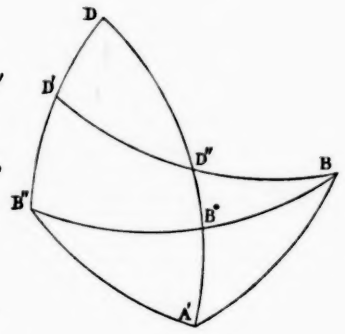
The first equation is obtained from equation III of the same article, by sub-

stituting the heliocentric positions of the earth for those of the celestial body whose orbit is to be determined; i. e. by putting $n = R'R'' \sin(l'' - l')$, $n' = RR'' \sin(l'' - l)$, $n'' = RR' \sin(l' - l)$, and $z = \delta'$.

The second equation is derived as follows: —

Put $L = \dfrac{\sin(A''D - \delta'')}{\sin(A'D - \delta' + \sigma)}$ (see equation 11) $= \dfrac{\sin B''D}{\sin B^*D}$. The triangle $B^*B''D$ gives

$$L = \frac{\sin B''B^*D}{\sin B^*B''D}.$$



From the triangles $A'B''B^*$, $A'BB''$, and $BB''D'$ we have

$$\sin B''B^*D = \sin B''B^*A' = \frac{\sin A'B''}{\sin A'B^*} \sin B^*B''A',$$

$$\sin B^*B''A' = \frac{\sin A'B}{\sin BB''} \sin BA'B'',$$

$$\sin B^*B''D = \frac{\sin BD'}{\sin BB''} \sin BD'B'';$$

whence $$L = \frac{\sin A'B''}{\sin A'B^*} \cdot \frac{\sin A'B}{\sin BD'} \cdot \frac{\sin BA'B''}{\sin BD'B''}.$$

A' is the heliocentric place of the earth (longitude l', latitude 0) at the time of the second observation; B and B'' are the geocentric positions of the celestial body at the times of the first and third observations (longitudes α, α'', latitudes β, β'').

Putting $$BA'B'' = BA'B^* + B^*A'B'',$$

and remembering that
$$\sin \beta = \sin A'B \cos BA'B^*,$$
$$\sin \beta'' = \sin A'B'' \cos B^*A'B'',$$
$$\cos \beta \sin(l' - \alpha) = \sin A'B \sin BA'B^*,$$
$$\cos \beta'' \sin(\alpha'' - l') = \sin A'B'' \sin B^*A'B'';$$

we have, after reduction,

$$\begin{aligned}\sin A'B'' \sin A'B \sin BA'B'' &= \sin \beta \cos \beta'' \sin(\alpha'' - l') - \sin \beta'' \cos \beta \sin(\alpha - l') \\ &= S \cos \beta \cos \beta'' \text{ (see art. 138. eq. 7)} \\ &= U \sin(\delta' - \sigma) \cos \beta \cos \beta''.\end{aligned}$$

We have also, by definition, $A'B^* = \sin(\delta' - \sigma)$, $BD' = AD' - \delta$, and $BD'B'' = \varepsilon'$;

$$\therefore \quad L = \frac{U \cos \beta \cos \beta''}{\sin(AD' - \delta) \sin \varepsilon'}, \quad Q.E.D.$$

[*Ormond Stone.*]

SOLUTIONS OF EXERCISES.

54

SHOW that if in a plane triangle ABC

$$\cos A + \cos B + \cos C = 1/2,$$

the centre of the circumscribed circle lies on the circumference of the inscribed circle. [*R. D. Bohannan.*]

SOLUTION.

The above equation may be written

$$1 + 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C = 1/2;$$

whence, squaring,

$$8 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C + 16 \sin^2 \tfrac{1}{2}A \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C - 1 = 0.$$

Dividing by $16 \cos^2 \frac{1}{2}A \cos^2 \frac{1}{2}B \cos^2 \frac{1}{2}C$, we have

$$\frac{\tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C}{2 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C} + \tan^2 \tfrac{1}{2}A \tan^2 \tfrac{1}{2}B \tan^2 \tfrac{1}{2}C - \frac{1}{16 \cos^2 \frac{1}{2}A \cos^2 \frac{1}{2}B \cos^2 \frac{1}{2}C} = 0.$$

Reducing this equation by means of the relations existing between R, r, and the angles of a triangle, we have

$$2Rr + r^2 - R^2 = 0,$$

or

$$R^2 - 2Rr = r^2.$$

But $1/(R^2 - 2Rr)$ equals the distance between the centres of the circumscribed and the inscribed circles; hence the proposition. [*Claude Waller.*]

74

FOUR points are taken at random on the surface of a sphere. What is the probability, that all of the points do not lie in the same hemisphere? [*A. Hall.*]

SOLUTION.

Let three points be taken at random on the surface of a sphere; they will all lie in the same hemisphere, and will form the vertices of a spherical triangle. If right lines be drawn from these points to the centre of the sphere, and extended to the surface, the points of intersection will be the vertices of a triangle opposite the first one. If the fourth random point is not in the same hemisphere as the first three points, it will be on the surface of the opposite triangle. Now the average area of this triangle is $\frac{1}{2}\pi r^2$, or one-eighth the surface of the sphere. Hence the required probability is $\frac{1}{8}$. [*A. Hall.*]

80

In exercise 4, what is the probability that the random circle exceeds the average circle? [*Artemas Martin.*]

SOLUTION.

Let x be the radius of the random circle; then from equation (1), solution of exercise 4, Vol. I. p. 67, we have

$$y = \sqrt{(2rx)}. \tag{1}$$

Let R be the radius of the average circle; then, p. 68,

$$\pi R^2 = \tfrac{1}{3}\tfrac{1}{2}\pi r^2\left[1 - \frac{\sqrt{3}}{\sqrt{3} + 2\log(\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{6})}\right];$$

whence

$$R = \tfrac{1}{4}r\left[\tfrac{1}{2} - \frac{\sqrt{3}}{2\sqrt{3} + 4\log(\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{6})}\right]^{\frac{1}{2}}. \tag{2}$$

If x be greater than R, the random circle will exceed the average circle. Putting R for x in (1) we get the inferior limit of y. The superior limit is $\frac{1}{2}r\sqrt{2}$, the same as in the solution of exercise 4.

Hence the required probability is

$$p = \int_{\sqrt{(2rR)}}^{\frac{1}{2}r\sqrt{2}} ds \div \int_{0}^{\frac{1}{2}r\sqrt{2}} ds$$

$$= \int_{\sqrt{(2rR)}}^{\frac{1}{2}r\sqrt{2}} (y^2 + r^2)^{\frac{1}{2}}\, dy \div \int_{0}^{\frac{1}{2}r\sqrt{2}} (y^2 + r^2)^{\frac{1}{2}}\, dy$$

$$= 1 - \frac{2\sqrt{(4R^2 + 2rR)} + 2r\log[\sqrt{(2R)} + \sqrt{(2R + r)}] - r\log r}{r\sqrt{3} + 2r\log(\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{6})}.$$

[*Artemas Martin.*]

81

FIND the equation of a curve whose ordinates represent the areas of the triangles in exercise 68. [*R. H. Graves.*]

SOLUTION.

The area of one of the triangles is

$$\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1) = \frac{1}{8a}\sqrt{(-27\Delta)},$$

where Δ = discriminant of the cubic,

$$y'^3 + (3a^2 - 4ax)y' - 8a^2y = 0.$$

$$\Delta = 64a^4y^2 + \frac{4}{3}(8a^2 - 4ax)^3$$
$$= 64a^4 \cdot 16a(x - 6a) + \frac{4}{3}(8a^2 - 4ax)^3,$$

by equation to locus, $y^2 = 16a(x - 6a)$. Whence

$$\frac{1}{8a}\sqrt{(-27\Delta)} = 2\sqrt{[a(x - 8a)^2(x + 10a)]};$$
$$\therefore \quad y^2 = 4a(x - 8a)^2(x + 10a)$$

is the required equation. It represents a parabola of the third degree having a loop and a node. The triangles correspond to the ordinates of those points for which $x > 6a$. [*R. H. Graves.*]

83

THE middle point of the segment AB is M. Find the locus of P when PM is a mean proportional between PA and PB. [*L. G. Carpenter.*]

SOLUTION I.

Assuming that AB is a straight line, let

$$AB = c, \ PM = m_c, \ PA = b, \ PB = a.$$

Equating 4 and 43 of Baker's *Formulæ for the Area of a Plane Triangle** we have

$$c = \sqrt{2(a^2 + b^2 - 2m_c^2)}$$
$$= \sqrt{2} \cdot (a - b),$$

since m_c is a mean proportional between a and b. Hence the locus of P is an hyperbola whose foci are A and B; centre, M; and eccentricity, $\sqrt{2}$.

[*Ormond Stone.*]

SOLUTION II.

$$AB^2 = 2PA^2 + 2PB^2 - 4PM^2$$
$$= 2PA^2 + 2PB^2 - 4PA \cdot PB;$$
$$\therefore \sqrt{2} \cdot MB = \pm (PA - PB);$$

therefore the required locus is an equilateral hyperbola.

[*R. H. Graves.*]

SOLUTION III.

Let $MP = r$; then, if x and y be the co-ordinates of P, we have

$$AP^2 = (a + x)^2 + y^2,$$
$$BP^2 = (a - x)^2 + y^2;$$

**Annals of Mathematics*, Vol. I. p. 136, and Vol. II. p. 12.

whence, by the conditions of the problem,

$$r^4 = (x^2 + y^2)^2 = [(a + x)^2 + y^2][(a - x)^2 + y^2];$$

$$\therefore\ x^2 - y^2 = \tfrac{1}{2}a^2,$$

a rectangular hyperbola, semi-major axis $a\sqrt{\frac{1}{2}}$. [*Ormond Stone.*]

84

FIND the thinnest frustum that can be cut from a given right circular cone by a plane parallel to the base, subject to the condition, that it may be laid on its slant surface on a horizontal plane without toppling over.

[*W. M. Thornton.*]

SOLUTION I.

Let X be the distance from the vertex of the cone to its centre of gravity, x_1 the distance to the centre of gravity of the cone that is cut off, and x the distance from the vertex to the centre of gravity of the frustum. Let M and m be the masses of the cones, and H and h their heights. For the centre of gravity of the frustum we have

$$\frac{x - X}{X - x_1} = \frac{m}{M - m},$$

or

$$x(M - m) = X(M - m) + (X - x_1)m.$$

Now

$$X = \tfrac{3}{4}H, \quad x_1 = \tfrac{3}{4}h;$$

and, since M and m are proportional to H^3 and h^3, we have

$$x(H^3 - h^3) = \tfrac{3}{4}H(H^3 - h^3) + \tfrac{3}{4}h^3(H - h),$$

or

$$x = \tfrac{3}{4}\left(H + \frac{h^3}{H^2 + Hh + h^2}\right).$$

If a be the angle between the axis of the cone and an element of its surface, the required condition gives

$$h = x\cos^2 a.$$

We have, therefore,

$$(4 - 3\cos^2 a)h^3 + (4 - 3\cos^2 a)Hh^2 + (4 - 3\cos^2 a)H^2h - 3H^3\cos^2 a = 0,$$

a complete cubic for h. The height of the frustum is $H - h$. If we put $H = 1$, we have

$$h^3 + h^2 + h - \frac{3\cos^2 a}{4 - 3\cos^2 a} = 0.$$

[*Asaph Hall.*]

SOLUTION II.

The line which joins the centre of gravity of the frustum to any point of the upper base must be perpendicular to the slant side; whence it follows that the

distance of this point from the upper base is

$$y = r \cdot \frac{R - r}{H - h} = \frac{Rr}{H} = h \tan^2 a.$$

The masses of the whole cone and of the frustum are as H^3 and $H^3 - h^3$; and the distances of their centres of gravity from that of the upper cone, $\frac{3}{4}(H - h)$ and $\frac{1}{4}h(1 + 4\tan^2 a)$;

$$\therefore \quad (H^3 - h^3) \cdot h(1 + 4\tan^2 a) = 3H^3(H - h);$$

whence, if we divide by $H - h$ and put $H = 1$, we get

$$h^3 + h^2 + h = \frac{3}{1 + 4\tan^2 a}.$$

[*W. M. Thornton.*]

[Mr. Geo. Eastwood sends a nearly equivalent solution.]

83

FIND the locus of the points from which the sum of the squares of the normals to the parabola $y^2 = 2px$ is constant.

SOLUTION.

Let x_1, y_1; x_2, y_2; x_3, y_3; x, y be the co-ordinates of the feet of the normals and their point of concourse. Then y_1, y_2, y_3 are the roots of the cubic in y',

$$y'^3 + 2p(p - x)y' - 2p^2 y = 0.$$

(Compare solution to exercise 68).

Then $\quad \Sigma[(x - x_1)^2 + (y - y_1)^2] = c^2$, a constant,

or $\quad 3(x^2 + y^2) - 2x\Sigma x_1 + \Sigma x_1^2 - 2y\Sigma y_1 + \Sigma y_1^2 = c^2,$

or $\quad 3(x^2 + y^2) - \frac{x}{p}\Sigma y_1^2 + \frac{1}{4p^2}\Sigma y_1^4 + \Sigma y_1^2 = c^2,$

or $\quad 3(x^2 + y^2) + \frac{x}{p}4p(p - x) + \frac{8p^2(p - x)^2}{4p^2} - 4p(p - x) = c^2;$

$$\therefore \quad x^2 + 3y^2 + 4px = 2p^2 + c^2,$$

which shows the required locus to be an ellipse. [*R. H. Graves.*]

91

An equilateral triangle, side $2a$, slides in a plane with one angular point on each of two rectangular axes. Find the locus of the third vertex.

SOLUTION I.

The motion is equivalent to that of a circle rolling inside another of double its diameter. The third vertex, being a carried point, describes an ellipse. De-

scribe a circle about the right triangle whose vertices are the sliding points and the origin, and draw a right line from its centre to the third vertex. Then from the points of intersection of the line and circle draw lines to the origin. These lines, which bisect the angles between the axes, give the positions of the axes of the curve. The semi-axes are equal in magnitude to the sum and difference of the altitude and half the side of the equilateral triangle, i. e. $a(\sqrt{3} \pm 1)$. (See Williamson's *Differential Calculus* [1884], Arts. 285 and 295.)

Remark.—The positions and magnitudes of the axes seem to be evident also from the symmetry of the figure. [*R. H. Graves.*]

SOLUTION II.

Let the base AB have its extremities A, B on Ox, Oy, and the vertex C will have for co-ordinates

$$x = 2a \sin(30^\circ + \varphi) = a(\cos\varphi + \sqrt{3} \cdot \sin\varphi),$$
$$y = 2a \cos(30^\circ - \varphi) = a(\sin\varphi + \sqrt{3} \cdot \cos\varphi);$$

if $AB = 2a$ and $BAO = \varphi$.

Eliminating φ, we get for the locus

$$x^2 - xy\sqrt{3} + y^2 = a^2,$$

an ellipse with its centre at O, and its axes in the bisectrices of xOy, whose lengths are

$$2a\sqrt{(4 \pm \sqrt{12})}. \qquad [\textit{Geo. Eastwood.}]$$

[More generally, for any triangle, if h be the altitude and a, b the segments into which it divides the base, the vertex is

$$x = a\cos\varphi + h\sin\varphi, \quad y = h\cos\varphi + b\sin\varphi;$$

and its locus is the ellipse,

$$(hx - ay)^2 + (hy - bx)^2 = (h^2 - ab)^2. \qquad W.\ M.\ T.]$$

SOLUTION III.

Let d be the abscissa of A; then

$$y^2 + (x - d)^2 = 4a^2,$$
$$d = 2a\cos\varphi;$$
$$\therefore \quad d = y\sqrt{3} - x,$$

and the equation of the locus is

$$x^2 - xy\sqrt{3} + y^2 = a^2,$$

an ellipse with its centre at the origin, axes inclined at angles of 45° to the co-or-

dinate axes, and semi-axes equal to

$$a\sqrt{(4 \pm \sqrt{12})}. \qquad [\textit{Frank Muller.}]$$

93

FIND the radius of the circle inscribed in the evolute of an ellipse, and the ratio in which each point of contact divides the quadrant on which it lies.

[*R. H. Graves.*]

SOLUTION I.

Let $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ be the equation to the ellipse;

then
$$\left(\frac{ax'}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{by'}{a^2 - b^2}\right)^{\frac{2}{3}} = 1$$

will be the equation to the evolute. Since from the form of the equation the centres of the evolute and the inscribed circle co-incide, the radius of the latter is equal to the minimum value of r'.

The problem thus resolves itself, practically, into the determination of the polar co-ordinates (r'_0, θ'_0) of the points of contact.

$$\xi = \left(\frac{a^2 - b^2}{a}\right)^{\frac{1}{3}} \cos\varphi, \quad \eta = \left(\frac{a^2 - b^2}{b}\right)^{\frac{1}{3}} \sin\varphi$$

are the co-ordinates of a point on the ellipse whose equation is

$$\left(\frac{a}{a^2 - b^2}\right)^{\frac{2}{3}} \xi^2 + \left(\frac{b}{a^2 - b^2}\right)^{\frac{2}{3}} \eta^2 = 1,$$

and $x' = \xi^3$, $y' = \eta^3$ satisfy the equation to the evolute.

It is evident that a minimum value of r' will correspond to a minimum value of

$$x'^2 + y'^2 = \xi^6 + \eta^6 = \left(\frac{a^2 - b^2}{a}\right)^2 \cos^6\varphi + \left(\frac{a^2 - b^2}{b}\right)^2 \sin^6\varphi.$$

Differentiating with regard to φ, this becomes

$$-6\left(\frac{a^2 - b^2}{a}\right)^2 \cos^5\varphi \sin\varphi + 6\left(\frac{a^2 - b^2}{b}\right)^2 \sin^5\varphi \cos\varphi;$$

whence dividing by $\quad -6(a^2 - b^2)^2 \sin\varphi \cos\varphi,$

and equating to zero, $\quad \frac{\cos^4\varphi}{a^2} - \frac{\sin^4\varphi}{b^2} = 0,$

or $\quad \tan^2\varphi = \pm\frac{b}{a}.$

The upper sign corresponds to the values

$$\xi_0 = \pm \left(\frac{a^2 - b^2}{a}\right)^{\frac{1}{3}} \left(\frac{a}{a+b}\right)^{\frac{1}{2}},$$

$$\eta_0 = \pm \left(\frac{a^2 - b^2}{b}\right)^{\frac{1}{3}} \left(\frac{b}{a+b}\right)^{\frac{1}{2}};$$

$$\therefore \quad x'_0 = \xi_0^3 = \pm (a-b) \left(\frac{a}{a+b}\right)^{\frac{1}{2}},$$

$$y'_0 = \eta_0^3 = \pm (a-b) \left(\frac{b}{a+b}\right)^{\frac{1}{2}};$$

$$\therefore \quad r'_0 = a - b,$$

$$\tan \theta'_0 = \pm \sqrt{\frac{b}{a}}. \qquad [\textit{Ormond Stone.}]$$

SOLUTION II.

At Fagnani's point on the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x^2 = \frac{a^3}{a+b} \quad \text{and} \quad y^2 = \frac{b^3}{a+b}.$$

Substituting in the values of the co-ordinates of the centre of curvature,

$$\alpha = \frac{(a^2 - b^2)\, x^3}{a^4} \quad \text{and} \quad \beta = -\frac{(a^2 - b^2)\, y^3}{b^4},$$

we obtain

$$\alpha = \frac{a^{\frac{1}{2}}(a-b)}{(a+b)^{\frac{1}{2}}} \quad \text{and} \quad \beta = -\frac{b^{\frac{1}{2}}(a-b)}{(a+b)^{\frac{1}{2}}};$$

$$\therefore \quad \alpha^2 + \beta^2 = (a-b)^2.$$

Also the perpendicular from the centre on the normal at Fagnani's point $= a - b$; $\therefore$ the circle $x^2 + y^2 = (a-b)^2$ touches the evolute at the point (α, β). The values of the radii of curvature at the ends of the axes are a^2/b and b^2/a; the radius of curvature at Fagnani's point $= 1/(ab)$; $\therefore$ the ratio of one segment of the quadrant to the other is

$$\left[\frac{a^2}{b} - 1/(ab)\right] : \left[1\ (ab) - \frac{b^2}{a}\right] = a^{\frac{3}{2}} : b^{\frac{3}{2}}.$$

[*R. H. Graves.*]

94

O is the centre of the circumscribed circle of ABC, and D, E, F the middle points of its sides. Show that

$$OI^2 + OE^2 + OF^2 = 2R'(2R' - r'),$$

where R', r' are the radii of the circumscribed and inscribed circles of the triangle of the feet of the altitudes. [*R. D. Bohannan.*]

SOLUTION.

Let A', B', C' be the feet of the altitudes, H their intersection point, G the intersection of the medians, and O' the circumcentre of $A'B'C'$, which circle contains also DEF.

It is well known that the radius of the circumcircle of ABC is $2R'$; whence we see that

$$OD^2 + OE^2 + OF^2 = 12R'^2 - \tfrac{1}{4}(a^2 + b^2 + c^2).$$

Again, since HD is the median of BHC,

$$HB^2 + HC^2 = \tfrac{1}{2}a^2 + 2HD^2;$$

and, in like manner,

$$HD^2 + HA^2 = GA^2 + GD^2 + 2GH^2$$
$$= \tfrac{5}{9}AD^2 + 2GH^2;$$

whence we find $\quad 2HA^2 + HB^2 + HC^2 = \tfrac{10}{9}AD^2 + 4GH^2,$

or, remembering that

$$AD^2 + BE^2 + CF^2 = \tfrac{3}{4}(a^2 + b^2 + c^2),$$

$$HA^2 + HB^2 + HC^2 = \tfrac{1}{3}(a^2 + b^2 + c^2) + 3GH^2,$$

or, since $\quad HA = 2OD, \quad HB = 2OE, \quad HC = 2OF,$

and $\quad GH = \tfrac{1}{3}O'H = \tfrac{1}{3}\rho,$

$$OD^2 + OE^2 + OF^2 = \tfrac{1}{12}(a^2 + b^2 + c^2) + \tfrac{1}{3}\rho^2.$$

From the first and last equations, we get

$$OD^2 + OE^2 + OF^2 = 3R'^2 + \rho^2$$
$$= 4R'^2 - 2R'r',$$

since $\quad \rho^2 = R'^2 - 2R'r'.$

[*W. O. Whitescarver.*]

95

In exercise 65 let k, l be the lengths of the perpendiculars CD, CE drawn at right angles to CA, CB to meet the base in D, E. Show that

$$\frac{m}{c} = \frac{k}{a} \cdot \frac{l}{b}.$$

[*Ormond Stone.*]

SOLUTION.

Equating two values for the double area of the triangles ABC, CDE, we

have, if h be their common altitude,

$$kl \sin C = hm,$$
$$ab \sin C = hc;$$

whence
$$\frac{kl}{ab} = \frac{m}{c}.$$

[*W. M. Thornton.*]

97

In the triangle ABC two lines drawn from C trisect the side AB. Given c, C, and the angle φ between the trisecants; to solve the triangle.

[*Marcus Baker.*]

SOLUTION.

Refer the figure to the base of the triangle as the x-axis of rectangular coordinates, with the origin at its middle point. Let xy be the vertex. The conditions of the problem give

$$x^2 + y^2 - cy \cot C = \tfrac{1}{4}c^2,$$
$$x^2 + y^2 - \tfrac{1}{3}cy \cot C = \tfrac{1}{36}c^2;$$

whence
$$y = \frac{2c}{3 \cot \varphi - 9 \cot C};$$

and the radius vector of the vertex is given by the relation,

$$\rho^2 = \frac{c^2}{12} \cdot \frac{3 \cot \varphi - \cot C}{\cot \varphi - 3 \cot C}.$$

By means of C and c draw the circumcircle, and intersect it with the circle whose radius is ρ. The points thus obtained give the desired solution.

[*W. M. Thornton.*]

101

Express in terms of $\sin^{-1} x$ and $\sin^{-1} y$

$$\tan^{-1} \frac{x + y}{\sqrt{(1 - x^2)} + \sqrt{(1 - y^2)}}.$$

SOLUTION.

If
$$\frac{x + y}{\sqrt{(1 - x^2)} + \sqrt{(1 - y^2)}} = \tan \theta, \quad x = \sin \varphi, \quad y = \sin \phi;$$

we may write
$$\tan \theta = \frac{\sin \varphi + \sin \phi}{\cos \varphi + \cos \phi};$$

whence
$$\sin (\theta - \varphi) + \sin (\theta - \phi) = 0;$$
$$\therefore \quad \theta - \varphi = \phi - \theta + 2n\pi, \quad \text{or} \quad \theta - \phi + (2n + 1)\pi;$$
$$\therefore \quad \theta = \tfrac{1}{2}(\varphi + \phi) + n\pi,$$

except when $\varphi = \phi \pm (2n + 1)\pi$; in which case θ is indeterminate.

[*J. L. Love.*]

102

FIND the relation connecting x, y, z when

$$\cot^{-1}(x + y + z - xyz) = \cot^{-1}x + \cot^{-1}y + \cot^{-1}z.$$

SOLUTION.

Adding $\frac{3}{2}\pi$ to each side, and changing signs, this becomes

$$\tan^{-1}(x + y + z - xyz) = \tan^{-1}x + \tan^{-1}y + \tan^{-1}z,$$

the tangent of which gives*

$$x + y + z - xyz = \frac{x + y + z - xyz}{1 - yz - zx - xy};$$

whence $x + y + z = xyz$, or $yz + zx + xy = 0$.

[*J. L. Love.*]

103

INTO a conical wine glass a spherical ball is dropped. Find the ratio of the concealed surfaces of the ball and the inside of the glass.

SOLUTION.

In the meridian section of the figure let C be the centre of the sphere, V the vertex of the cone, AB the chord of contact, and H, D the intersections of AB and the sphere with CV.

The conical surface is $c = \pi \cdot HB \cdot VB$

$$= \pi \cdot HV \cdot CD,$$

since, in virtue of the similarity of VBH, VCB,

$$HV : VB = HB : (CB = CD).$$

The spherical surface is $s = 2\pi \cdot CD \cdot HD.$

Hence the required ratio is $\frac{s}{c} = \frac{2HD}{HV}$. [*O. L. Mathiot.*]

*See Chauvenet's *Trigonometry*, p. 39. eq. (170).

EXERCISES.

104

A SPIRAL of Archimedes having its pole in the circumference of a given circle, passes through the centre and the point 120° of the circumference from the pole. Find its equation. [*O. Root, Jr.*]

105

IF the centre of the circumscribed circle of a triangle is on the circumference of the inscribed circle,

$$\cos \tfrac{1}{2}A \cdot \cos^2 A + \cos \tfrac{1}{2}B \cdot \cos^2 B + \cos \tfrac{1}{2}C \cdot \cos^2 C = 0.$$

If the same point is on the circumference of one of the escribed circles, a like relation holds. [*W. M. Thornton.*]

106

IF m be a positive integer,

$$\sin (m-1)\varphi - x \sin m\varphi + x^m \sin \varphi$$

will contain $1 - 2x \cos \varphi + x^2$. [*A. Hall.*]

107

GIVEN one vertex of a rectangle and the ratio of its sides, construct the rectangle so that the extremities of the diagonal opposite to the given vertex shall lie

1. On two given parallel straight lines,
2. On two given intersecting straight lines,
3. One on a given line and the other on a given circle,
4. On two given circles. [*R. D. Bohannan.*]

108

CONSTRUCT an equilateral triangle of given size inscribed in a given equilateral hyperbola. [*R. H. Graves.*]

109

EACH of four equal circles touches two of the others. Find the average area included between them. [*Artemas Martin.*]

SELECTED.

110

SOLVE the triangle ABC; given a, A and the product $\beta\gamma$ of the bisectrices drawn from B, C to CA, AB.

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